DOCUMENT RESUME

ED 211 143 JC 810 647

AUTHOR Lay, L. Clark
TITLE An Honors Approach to Remedial Instruction.
PUB DATE 9 Oct 81
NOTE 21p.: Paper presented at the National Convention of the American Mathematical Association of Two-Year Colleges (New Orleans, LA, October 7-11, 1981).

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS Academic Achievement; *Algebra; Community Colleges; *Mathematical Concepts; *Remedial Instruction; *Remedial Mathematics; Student Placement; *Teaching Methods; Two Year Colleges

ABSTRACT

A rationale is presented for an honors approach to remedial algebra instruction. Beginning with questions to be considered in the initial placement of students, the paper suggests that remedial students be enrolled at a level low enough to provide them with a reasonable chance to be good, even honors, students. It goes on to identify and contrast two approaches to mathematics instruction, Advanced Mathematics from an Elementary Standpoint (AMES) and Elementary Mathematics from an Advanced Standpoint (EMAS), and concludes that EMAS is more appropriate for remedial classes. The paper then suggests beginning instruction at the 4th-grade level, an achievement level common to remedial students, and enumerates the algebraic concepts accessible to students at this level. Several broad assumptions are then enumerated concerning remedial mathematics instruction, e.g., the low mathematical competence among students, the failure of past curricular reforms, sources of student difficulties in mathematics, the appropriate sequencing of concepts in instruction, and the need for content improvement in courses. In the remainder of the paper, three examples of using EMAS methods to teach the transformation and solution of linear equations are presented and illustrated. These examples illustrate the analysis of linear equations using the monoids for addition and multiplication and the partial groupoids for subtraction and division which are basic math competencies at the 4th-grade level. (KL)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED211143

AN HONORS APPROACH TO REMEDIAL INSTRUCTION

L. Clark Lay

Paper presented at the National Convention of the
American Mathematical Association of Two-Year Colleges
(New Orleans, LA, October 7-11, 1981).

JC 810 647

U.S. DEPARTMENT OF EDUCATION
NATIONAL INSTITUTE OF EDUCATION
EDUCATIONAL RESOURCES INFORMATION
CENTER (ERIC)

X This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

• Points of view or opinions stated in this document do not necessarily represent official NIE position or policy.

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY
L. Clark Lay

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

An Honors Approach to Remedial Instruction
A paper presented at the National Convention of the
American Mathematical Association of Two-Year Colleges
Oct. 9, 1981 by L. Clark Lay

The title of these remarks may seem self-contradictory. Our purpose is to support it as a characterization of a philosophy of remedial instruction that differs in significant ways from the philosophy that has guided past practice, and from that which remains in current acceptance. The application will be limited to remedial algebra, a course offering that is now listed in the remedial programs of nearly all colleges.

Just now, think of yourself as the mathematics department chairman in a two year college as you counsel a prospective student. A large chart hangs on the office wall. It is a flow diagram that lists your course offerings, beginning with arithmetic and working up to differential equations and linear algebra. Its purpose is to show the sequential order for these courses, and to reveal at a glance the necessary pre-requisites for each of them.

There has been some explanation to the student of the nature of these courses, and his or her past mathematics background has been reviewed. You are about to advise this student concerning the proper course in which he should enroll. Before making a final decision, you ask this question: Suppose time and expense is not a consideration, but getting an "A" is an absolute must. In what course would you enroll?

Many of my students have found this question to be quite disturbing. After all, if one has never received an A in mathematics, at what level should one choose to study, in order to maximize the chance of becoming an A student? If one begins too high the task will be impossible. But if the choice is too low then boredom and lack of challenge will also guarantee failure.

Why should a mathematics counselor ask such a question? The answer may lie in our response to some further questions. Have you ever noticed a tendency for students in your classes to separate into A's and F's? Does your experience verify the old saying, "The rich get richer, and the poor get poorer"?

Let us reformulate this conjecture so that you can test it in your own curriculum by checking, for example, student records over three consecutive terms of calculus. Do your A students have a high probability of getting A's in the next sequential course? Do B students have a somewhat less chance of getting a B or more in the next course? And do the C students have a much lower probability of earning at least a C?

This principle may have some general application. Even the Bible makes note of it. In Matthew 13:12 (King James Version) we read: "For whosoever hath, to him shall be given, and he shall have more abundance; but whosoever hath not, from him shall be taken away even that he hath." How then should we plan for the remedial or developmental student, so that he or she can avoid the vicious downward spiral and join the golden spiral that leads upward?

The first decision can be one of enrolling the remedial student at a level low enough to provide him or her with a reasonable opportunity for behaving like a good student, or even as an honor student. Next, what course content should be offered? The conventional topics (for the remedial student, the same old stuff) hold little promise. Some novelty must be sought, whatever be the price.

The paradox now confronts us. How can we ever expect concepts from a high level to be accessible and profitable for a student with a record of failure at a much lower level? The answer may lie in a point of view.

As an illustration of such a viewpoint consider one of the classics of mathematical exposition, Felix Klein's Elementary Mathematics from an Advanced Standpoint, first written in 1908. Prof. Klein was a world renowned researcher and teacher. He was lecturing at the University of Gottingen, at that time the acknowledged summit of the mathematical world. The book was a transcription of his lectures to doctoral students, who as per German custom at that time, were soon to begin their teaching career as instructors in the Gymnasia (secondary schools). This explains his choice of title.

But suppose the reader is (as I was) a teacher in a U.S. high school with an only then emerging interest in mathematics, and having only a BA minor in mathematics? Even though I read this book with great interest, and after a time with much profit, it soon became apparent that this was not for me, Elementary Mathematics from an Advanced Standpoint. Quite the opposite. It was Advanced Mathematics from an Elementary Standpoint!

The differences in these two viewpoints of mathematics seem essential to our concept of an honors approach to remedial mathematics. The assumptions listed below will be the guidelines. Consider which of these you can, at least tentatively, accept. As mathematicians we can hardly expect to agree on the conclusions to be drawn unless we start with some of the same postulates.

Two Views of Mathematics

(AMES) Advanced Mathematics from an Elementary Standpoint
contrasted with
(EMAS) Elementary Mathematics from an Advanced Standpoint

(AMES) A first look at unfamiliar ideas
(EMAS) A second look at familiar ideas

(AMES) Easy questions about hard (?) concepts
(EMAS) Hard questions about easy (?) concepts

(AMES) Higher initial interest, but motivation decreases with time as the novelty wears off
(EMAS) Lower initial interest, but motivation increases with time as the sense of mastery grows

(AMES) Novel treatment of topics is less essential, since the topics are novel in themselves
(EMAS) Novel treatment of topics is very essential, since the topics are familiar

(AMES) Minimum variety of developments and techniques can be introduced for complex and unfamiliar ideas

(EMAS) Maximum variety of developments and techniques can be introduced for simple and familiar ideas

(AMES) Mathematical maturity through awareness of the scope and applicability of mathematics

(EMAS) Mathematical maturity through critical thinking and refinement of intuition

(AMES) Small increase in problem solving power from the necessarily introductory treatment of topics

(EMAS) Greater increase in problem solving power from the deeper treatment possible for topics

(AMES) Increase in the number of mathematical concepts known superficially and with little confidence

(EMAS) Increase in the number of mathematical concepts known thoroughly and with great confidence

(AMES) Little increase in computational and manipulative skills

(EMAS) Significant increase in computational and manipulative skills

(AMES) Minimum effectiveness for use in remedial classes

(EMAS) Maximum effectiveness for use in remedial classes

(AMES) Students handicapped by underconfidence
They think they cannot succeed

(EMAS) Students handicapped by overconfidence
They think they know already

(AMES) Prerequisite for Teacher: Interest in gaining knowledge of more advanced mathematics

(EMAS) Prerequisite for Teacher: Interest in better understanding of elementary mathematics

Some find learning relatively easy; others find it difficult to learn; for still others, learning is not possible.

Those with limited mental capacity, particularly for symbolic representation, space perception, and deductive reasoning, will have limited success in mathematics. This is especially true for AMES.

Those who never learn are those that think they know already. This can close the mind to EMAS.

To illustrate these contrasting views we consider some basic algebraic systems as presented in Fig. 1. Variations of such diagrams can be found in texts on abstract algebra. An algebraic system here refers to a complex: A domain set, one or more operations (unary, binary, n-ary), or one or more relations defined on the domain set. The hierarchy as in Fig. 1 results as one assumes certain properties for the operations or relations (well-defined, associative, etc.), or when one identifies certain distinguished members of the set such as an identity.

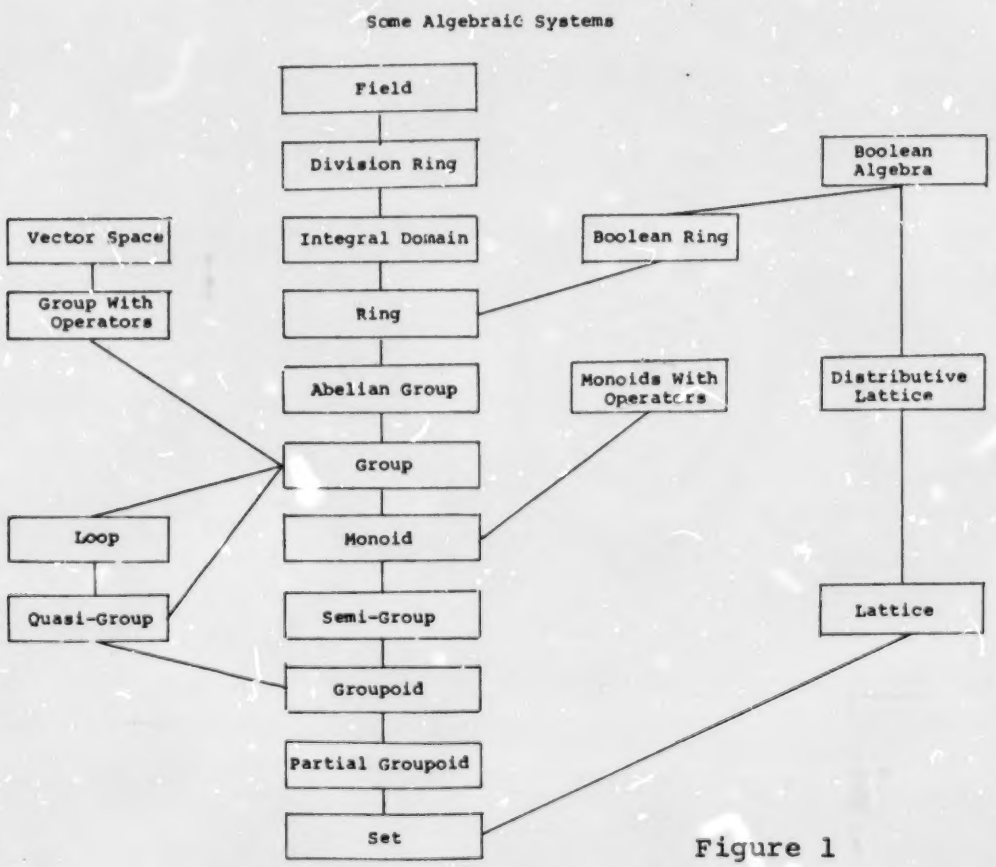


Figure 1

It appears fair to state that such structures were presented as Advanced Mathematics from an Elementary Standpoint to the secondary teachers who enrolled in the many National Science Foundation Institutes of the 1960's. A limited selection of these systems, such as groups, rings, and fields, were introduced to teachers who, for the most part, had little prior knowledge of them. The explanatory examples did include some that the teachers could, if desired, present in turn to their own students. Much of the material that was developed in these institutes was later published as articles, monographs, and texts. With these as our guide the study of algebraic systems is in this case seen to have mainly a motivational character. Thus, it did not result in a substantial revision of the secondary mathematics curriculum.

Much of modern reform in elementary mathematics, including the so-called "new" mathematics, was indeed held out with the promise of being elementary mathematics from an advanced standpoint. I raise this question: Was this promise fulfilled, or did we, as suggested above, get instead advanced mathematics from an elementary standpoint? If so, we want to avoid making the same mistake. Hence we are <u>not</u> suggesting a "baby" course in abstract algebra, as being appropriate for remedial students.

But, if time permitted, we would want to deal with monoids, monoids with operators, quasi-groups, groups, groups with operators, and vector spaces. Where in elementary mathematics should we look for such structures? Why not in grade 4? After all, a number of entering students at many colleges, including mine, cannot score higher than grade 4 on a standardized grade level test in arithmetic. So if I am to contact all of my students at the common sense level I can begin no higher that grade 4.

Fortunately, some of the structures just mentioned are already there, begging for recognition. And with some slight adjustments the rest can become readily accessible without moving above this grade level. What is the mathematical structure in grade 4, as viewed from above with the advantage of an algebraic standpoint?

The number domain is the set of non-negative integers.
(At this point, no fractions or negatives)

Addition is a binary operation over this domain.
(Additive groupoid)

Subtraction is a partial binary operation over this domain.
(Partial groupoid, defined over a proper subset)

Addition is associative.
(Additive semi-group)

Zero is the identity for addition.
(Additive monoid)

The group structure is lacking.
(No inverse elements)

Addition over this set forms a commutative monoid.

Subtraction is neither commutative nor associative and lacks an identity.

Similarly, multiplication over the set of non-negative integers yields a multiplicative commutative monoid, with 1 as the identity.

Division over this set yields only a partial groupoid.

The technical language used above may be appropriate for college teachers of mathematics, but this is certainly not the case for fourth graders. And it could be forbidding even for college students, if their mathematical stature is only at this level. What then is the pay-off for all this fanfare? It is time for some classroom examples. But first we state the broad assumptions that we make, even though many must remain without further comment at this time. They are taken from a paper with this title given at the National Conference on Remedial and Developmental Mathematics, sponsored by the City University of New York, in New York City, April 9-11, 1981. The proceedings of this conference are now in preparation for publication.

Semigroups, and Semigroups with Operators, in Remedial Algebra.

Some Assumptions

There is now a serious crisis in mathematical education. The majority of students in our schools are failing to attain the mathematical competence necessary to advance in their chosen careers.

The curriculum reforms of the past decades have been disappointing. There has been little success in solving the major problem: the need for a better understanding of the simple concepts in mathematics in order to make them accessible to more people.

A significant portion of the difficulties encountered in remedial mathematics stems from concepts that are never explained, or even mentioned. Conjecture: These concepts are considered to be obvious by teachers, authors of texts, and the best students.

The optimum laboratory for development of better elementary mathematics can be located in the remedial programs of the secondary schools and colleges.

The historical order of positive integers, non-negative integers, non-negative rationals, rationals, irrationals, and then reals should guide the extensions of number in the curriculum.

Composition of dynamic phenomena should precede or accompany combination of static phenomena.

> Increases and decreases before negative numbers.
> Expansions and contractions before rational numbers.
> Compositions of changes (functions) as well as binary operations on numbers.

The present curriculum lacks "readiness" for the extension of the number system. In the domain of the non-negative integers:

> Sums and differences can be studied in a way that makes the properties of signed integers familiar <u>before</u> the number systems is extended to include them.
>
> Products and quotients can be studied in a way that makes the properties of the non-negative rationals familiar <u>before</u> the number system is extended to include them.

The study of structures with inverses (groups, rings, and fields) should be paralleled by, and at times preceded by, the study of structures without inverses (groupoids, semigroups, monoids).

The study of binary operations should be paralleled by, and at times preceded by the study of unary operations.

Transformations and solutions of linear equations can begin with the axioms for additive and multiplicative semigroups, then addition and multiplication operators on these semigroups can be introduced, and finally the conventional field axioms can be used.

Mathematics education advances along two fronts: Content and methodology. The greatest promise for success lies in better mathematical content.

There should never be conflict between rigorous mathematics and effective pedagogy, provided we choose the right mathematical system to model for the learner.

The growing edge of mathematical research is found mainly in advanced topics. Concern for better understanding of the simple things is stimulated most during periods of crisis.

As an example of the conclusions that follow from one of the above assumptions, let the topic be the transformation and solutions of linear equations, which is one of the first major tasks undertaken in the beginning study of algebra. The algebraic structure that underlies the algorithms for such solutions and changes of form has had very little change over the past several centuries. The student has always needed essential skills with signed numbers (additive inverses) and fractions (multiplicative inverses). Just as important, although somewhat more subtle, he or she must be familiar with the concept of equivalence. For there can be little intuitive grasp of such much used phrases as "equals added to equals" if the very idea of "equals" is not well understood.

When we check against the algebraic systems that are related in Fig. 1, we can note the complexities that confront the beginning algebra student. He or she must operate within a field, and this is the most highly structured of all the systems shown.

But the analysis of linear equations can be undertaken in a much simpler and less sophisticated setting. For as noted above, at the fourth grade there are already at hand the monoids for addition and multiplication, and partial groupoids for subtraction and division. The utilization of these resources has been completely overlooked, and is now long overdue in mathematics instruction.

One modification of the usual curriculum at this level is needed. This is the introduction of the form $\frac{x}{y}$ for the division of x by y, and the use of this form prior to the later use of $\frac{x}{y}$ to represent fractions and ratios. For example, for 8 divided by 2, the division algorithm of $2\overline{)8}$ is clearly not appropriate for use with equations. The form $8 \div 2$ is also avoided for several reasons. While parentheses are redundant for $(8) \div (2)$ and are omitted, an excessive use of parentheses does become necessary for more complex forms. By contrast, $\frac{8}{2}$ is read as though it were $\frac{(8)}{(2)}$, so that the bar of this notation has the effect of a built-in parentheses. Furthermore, the form $8 \div 2$ tends to conceal and thus hinder the identification that comes later of the common properties of quotients, fractions, and ratios.

The discussion to follow in Example 1 is offered as an instance of elementary mathematics from an advanced standpoint. It is given as it could be presented to remedial students having only fourth grade achievement levels, but by utilizing the resources of semi-group theory. As such, it is an important component of our honors approach to remedial algebra.

Both college and secondary teachers of mathematics can recognize the essential nature of the skills involved, when approached in the conventional manner. Unfortunately, it is easily verified that many students have not acquired these skills after two years of algebra instruction. We suggest, as an alternative, some quite different thought patterns.

Example 1. To transform $a - \frac{b}{c} = d$ to forms that are solved explicitly for the variables a, b, and c. (It is now solved explicitly for d.)

Note: The assumed number domain is that of the non-negative integers: 0, 1, 2, 3, ...
In this context, the form $a - \frac{b}{c}$ is not defined unless c divides b, and it is further required that a is greater than or equal to $\frac{b}{c}$.

Numerical examples that meet these conditions are easy to construct, such as $9 - \frac{8}{2} = 5$. The necessary supplementary instruction with the inequality and divides relations is encouraged rather than avoided, since these concepts are very valuable in themselves. All mathematics students at this stage should not defer learning that:

$a > b$, if and only if there exists a unique positive number n such that $a = b + n$

$a|b$, if and only if there exists a unique number m, such that $mb = a$

Note: Until fractions are introduced, m must be an integer.

Example 1.(continued)

$a - \frac{b}{c} = d$ $\qquad$ $9 - \frac{8}{2} = 5$ $\qquad$ (A)

$a - (\frac{b}{c}) = d$ $\qquad$ $9 - (\frac{8}{2}) = 5$ $\qquad$ (B)

If we think (or at first write) as in (B), we can observe a sum - addend relationship of the form $S - a_1 = a_2$, where it can be implied that S is the sum, with a_1 and a_2 as two addends. This relationship is then made explicit.

$a = d + (\frac{b}{c})$ $\qquad$ $9 = 5 + (\frac{8}{2})$ $\qquad$ (C)

We now have a form that is solved explicitly for the variable a. To proceed, return to (B) above, and reflect that if $S - a_1 = a_2$ then it must be that $S - a_2 = a_1$, and hence that $a_1 = S - a_2$

$(\frac{b}{c}) = a - d$ $\qquad$ $(\frac{8}{2}) = 9 - 5$ $\qquad$ (D)

Note: The conventional statement of the commutative property of addition is applied to a binary form, such as $a_1 + a_2 = a_2 + a_1$. A useful extension of this concept is: <u>Th addends of a sum can be commuted</u>. Hence, when in (B) the numbers $(\frac{8}{2})$ and 5 can be recognized in $9 - (\frac{8}{2}) = 5$ as being addends for the sum 9, these addends can be interchanged to give $9 - 5 = (\frac{8}{2})$.

Example 1. (continued)

The change from (A) to (B) was one of recognition of a relation connecting three numbers: a, $(\frac{b}{c})$, and d. The equation forms in (B), (C), and (D) are explicit, respectively, for d, a, and $(\frac{b}{c})$.

Some essentially different relations will be evident only if three different numbers can be identified. This is not possible for equation forms (B) and (C), but there is this opportunity with (D). We can stop thinking (or writing) $(\frac{b}{c}) = a - d$, and shift to thinking (or writing) $\frac{b}{c} = (a - d)$.

$\frac{b}{c} = (a - d)$ $\qquad$ $\frac{8}{2} = (9 - 5)$ $\qquad$ (E)

The equation form, (E), reveals three different numbers: b, c, and (a - d). Furthermore, these new numbers are in a product-factor relationship, $\frac{P}{f_1} = f_2$. Here b is the product P, and c and (a - d) are the factors, f_1 and f_2. The next step can be an

explicit statement for the product, P.

$b = c(a - d)$ $\qquad\qquad$ $8 = 2(9 - 5)$ $\qquad\qquad$ (F)

From (F) we can reason that if $\frac{P}{f_1} = f_2$, then it must be that $\frac{P}{f_2} = f_1$. The commutative property of multiplication can be restated as: The factors of a product can be commuted. Then, from (E) we have:

$\frac{b}{(a - d)} = c$ $\qquad\qquad$ $\frac{8}{(9 - 5)} = 2$ $\qquad\qquad$ (G)

and then that

$c = \frac{b}{a - d}$ $\qquad\qquad$ $2 = \frac{8}{9 - 5}$ $\qquad\qquad$ (H)

Example 1. Summary

Each of the variables a, b, c, d has now been exhibited in its explicit relationship to the other three variables. See (A), (C), (F), and (H).

The reasoning used in Example 1 can be very informal, since it is based on ideas that have been accepted by the students for a considerable period of time. Thus if $5 = 3 + 2$, then surely $3 = 5 - 2$, and $2 = 5 - 3$. And again, since $6 = 3 \cdot 2$, then $3 = \frac{6}{2}$, and $2 = \frac{6}{3}$.

The application of these principles, as in Example 1, requires familiarity with the basic forms that are set out in the diagram for the next example.

Example 2

Equivalent Sums and Differences

(1)	$S = a_1 + a_2$	$S = a_2 + a_1$	(2)
(3)	$a_1 + a_2 = S$	$a_2 + a_1 = S$	(4)
(5)	$S - a_1 = a_2$	$S - a_2 = a_1$	(6)
(7)	$a_2 = S - a_1$	$a_1 = S - a_2$	(8)

Equivalent Products and Quotients

(9)	$P = f_1 \cdot f_2$	$P = f_2 \cdot f_1$	(10)
(11)	$f_1 \cdot f_2 = P$	$f_2 \cdot f_1 = P$	(12)
(13)	$\frac{P}{f_1} = f_2$	$\frac{P}{f_2} = f_1$	(14)
(15)	$f_2 = \frac{P}{f_1}$	$f_1 = \frac{P}{f_2}$	(16)

Figure 2

Equations 1-4 in Fig. 2 identify the sum S explicitly. But equations 5-8 are implicit for S, or as we say, these equations are not solved for S. The relation between equations 3 and 5, or between 4 and 6, can be given by definition.

Definition: Subtraction in terms of addition.
$a - b = c$ if and only if $b + c = a$

A first number a minus a second number b is the difference c if and only if the second number b plus the difference c is the first number a.

Again, equations 9-12 in Fig. 2 are explicit in form for the product P. But the recognition of P as the product is only implied by equations 13-16, since these equations are not solved for P. The relation between equations 11 and 13, or between 12 and 14, can be given by definition.

Definition: Division in terms of multiplication.
If $b \neq 0$, then $\frac{a}{b} = c$ if and only if $b \cdot c = a$

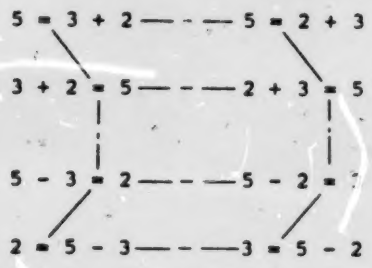
A first number a divided by a second number b is the quotient c if and only if the second number b times the quotient c is the first number a.

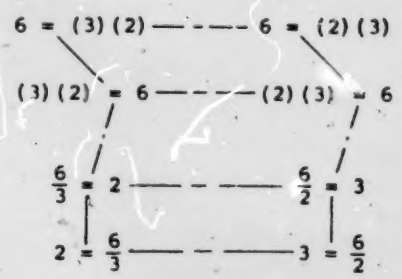
Students can readily learn to recognize the sum and to distinguish it from the addends, in either the explicit or the implicit equation forms. Because sums are commutative, the distinction between the odd and even numbered equations is arbitrary, and need not be emphasized. Hence, in practice, there are only 8 and not 16 forms to be recognized.

We underscore our belief that the logical structure underlying these equivalent equations should not be used as an introduction to their study. It is rather to be an anchor in the mind of the teacher, and to serve as a guide toward the eventual understanding to be achieved by the students.

The simplicity of this logical structure is shown for some example equations by the diagrams for Example 3. The upper two diagrams are coded to show the use of the assumptions needed from the addition and multiplication monoids of Grade 4. (See Figure 1.)

Example 3 Equivalent Equations



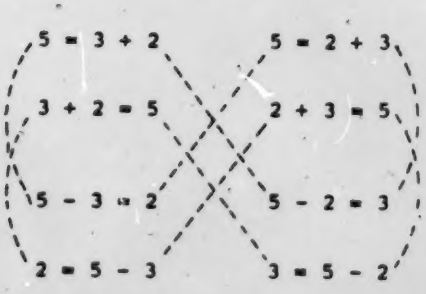


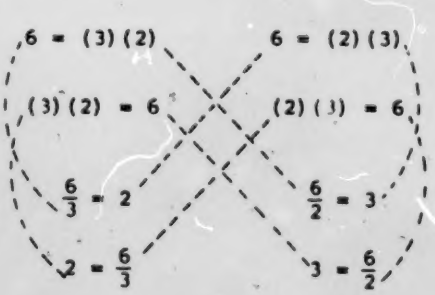
Code: ——— Symmetric property of Equality

— — — Commutative Properties

— · — Definition of Subtraction, or Division

- - - - - Removing and Joining Inverse Operators





When the time arrives for emphasis on the logic, the central feature can be the definitions of subtraction and division. Definitions, as such, are given little attention in the lower grades. Here is an opportunity to stress that much of the power of mathematics rests on precision of statement, and that mathematicians are serious in their efforts to say what they mean, and to mean what they say, no more and no less.

The symmetric property of equality is also almost entirely ignored at the elementary level, and will need some attention. In particular, the symmetry of the two members of an equation must be distinguished from the commutativity of two addends of a sum, since both changes of form involve an interchange of symbols.

The two lower diagrams for Example 3 are relevant to the next stage that we are suggesting for the study of equivalent linear equations. The new concept is one which Bourbaki, in the introduction to Algebra I, so aptly refers to as a "law of action".[1]

Consider the equation, 3 + 2 = 5, with attention directed to the left member, 3 + 2. It is customary to think of this as a binary form, $\underline{\underline{3 + 2}}$, with 3 as a first number, 2 as a second, and the plus sign as a link which forms a third number, 3 + 2. This equation is a true statement since the equals sign joins two names for the same number, five.

But one can also, without change of symbolism, think of 3 + 2 with a unary operation point of view. This time think $\underline{\underline{3 + 2}}$. The first number is still 3, but the + 2 is not a numeral for a number; it represents a transformation, an increase of 2. The number 3 is the operand, or initial state; the operator is + 2; and the transform, or final state, is 3 + 2.

When the right operator + 2 is written at the right of a number, such as 3, the joining of this operator results in an increase of 2. But when the original state is 3 + 2, then the removing of the operator + 2 results in a decrease of 2. Hence the lower left diagram links the equation 3 + 2 = 5 (five equals

five) with the equation 5 - 2 = 3. (Three equals three.)

This equation transformation is one of "doing the same thing to both members" only in the sense of making the same change to both numbers. What the eye observes are two quite different physical acts. In one member the decrease of two is produced by erasing + 2; in the other member the decrease of two is made by writing - 2. The operators, + n and - n, are inverse operators; joining one is equivalent to removing the other.

At this point experienced teachers will recognize a "transposition" manuever, well known as a valued friend of the good student, but a mortal enemy of a poor one. This is a unary operation device. One does something to a first number to change it into a second. By using operators, transposition can be accepted into the polite society of mathematical discourse.

But we can also think of 3 + 2, as $\underline{3 + 2}$. This time 2 is the operand, and 3 + is the left operator that represents an increase of 3. Hence the diagram also links 3 + 2 = 5 to 2 = 5 - 3. (3 + is removed, - 3 is joined.) No attempt is made at this time to define 5 - as operator. Such a change is more complex and is not needed at this point.

Pairs of inverse operators exist in considerable variety throughout mathematics. In the lower right diagram of Example 3 the multiplication operators, n( ) and ( )n, and the division operator $\frac{( )}{n}$ are used. An important observation about inverse operators is that members of different families don't mix. One must not expect to "undo" + 2 by writing $\frac{( )}{2}$. Working with operators proves to be a powerful antidote for another classroom poison known as "unlimited cancellation". A rationale is provided so that students no longer expect $\frac{8 + 2}{2}$ to equal 8.

Another reason why thinking operators is essential to correct cancellation is that only operators ever cancel; $\underline{\text{numbers never do}}$. Try cancelling the twos in $\frac{2}{2 \cdot 3}$. Clearly the nonsense form of $\frac{}{\cdot 3}$ is not what is wanted. But if the identity operator $\cdot 1$ is introduced to get $\frac{2 \cdot 1}{2 \cdot 3}$, then both the operators, $2 \cdot$, can be recognized as having an inverse role and then cancelled to get $\frac{1}{3}$ as desired.

When operator thinking is used with equations it is also essential to recognize just which operators are accessible for removal. The answer is: The last operator joined in the construction of an expression (following the conventional order of operations) is the first to be removed.

For the equation, $a - \frac{b}{c} = d$, in Example 1, the operator $\frac{(\ )}{c}$ is not accessible for removal from $a - \frac{b}{c}$, since division by c is <u>not</u> the last operation to be used in evaluating $a - \frac{b}{c}$. The only accessible operator in this form is $- \frac{b}{c}$. Hence from $a - \frac{b}{c} = d$, we obtain $a = d + \frac{b}{c}$.

In the form $a = d + \frac{b}{c}$, or in $d + \frac{b}{c} = a$, the operator $d +$ is now accessible. Hence we have $\frac{b}{c} = a - d$, by removing $d +$, and joining $- d$. In $\frac{b}{c} = a - d$, the operator $\frac{(\ )}{c}$ is accessible, so the next equation transformation can be to $b = c(a - d)$. Here $\frac{(\ )}{b}$ is removed, and $c(\ )$ is joined.

Space does not permit completion of this discussion of equivalent linear equations. One important omission is the use of the distributive laws, such as replacing $ab + ac$ with $a(b + c)$ to provide for the so-called combining of like terms. Equally vital is a study of sums and differences that makes it intuitively acceptable that $a - (b - c)$ can, on occasion, be replaced by $a - b + c$ or $a + c - b$. There is a price to be paid for this, however, since we must be careful in choosing a, b, and c so that all differences are defined.

But the price is small, since there is a double bonus. We have opportunity to become familiar with inequalities in a very elementary setting. And also, as stated above in our assumptions, we can become familiar with all the properties of signed integers <u>before they need ever be mentioned by this name</u>.

The physical "action" interpretations for addition and subtraction operators are increases and decreases, respectively. For multiplication and division operators the dynamic models can be expansions and contractions. The transformation above of $a - (b - c)$ to $a - b + c$ or $a + c - b$ is a change of order of

operations. By an analogy, addition to multiplication and subtraction to division, we arrive at the problem of "simplifying" the complex form $\frac{a}{\frac{b}{c}}$. Fortunately, it can be shown that there is a way of thinking about expansions and contractions that makes it quite plausible (under the proper conditions) for this form to be equivalent to $(\frac{a}{b})c$, or to $\frac{ac}{b}$. Again, since fractions are not as yet available, the choice of a, b, and c is quite restricted if all quotients are to be defined.

But the study of the necessary divides relations is precisely the analysis that is essential to an understanding of complex fraction forms. Such studies of products and quotients make it possible for the student to become familiar with all the properties of fractions before they need ever be mentioned by this name. This is what we mean by "readiness" for extensions of the number system, as examples of an honors approach to remedial mathematics.

Bibliography

1. Bourbaki, (1974) Algebra, Part 1, Addison-Wesley, Reading

JAN 22 1982

ERIC Clearinghouse for Junior Colleges
96 Powell Library Building
University of California
Los Angeles, California 90024